

# Effective geometries and generalized uncertainty principle corrections to the Bekenstein–Hawking entropy

Ernesto Contreras

*Centro de Física Teórica y Computacional, Facultad de Ciencias,  
Universidad Central de Venezuela, AP 47270, Caracas 1041-A, Venezuela.*

Fabián D. Villalba and Pedro Bargueño\*

*Departamento de Física, Universidad de los Andes,  
Apartado Aéreo 4976, Bogotá, Distrito Capital, Colombia*

In this work we construct several black hole metrics which are consistent with the generalized uncertainty principle logarithmic correction to the Bekenstein–Hawking entropy formula. After preserving the event horizon at the usual position, a singularity at the Planck scale is found. Finally, these geometries are shown to be realized by certain model of nonlinear electrodynamics, which resembles previously studied regular black hole solutions.

## I. INTRODUCTION

Black hole (BH) entropy can be considered as the paradigmatic quantum gravitational effect *par excellence* one can think of. After the initial findings by Bekenstein [1–3], Hawking realized [4, 5], within the framework of quantum field theory in curved backgrounds, that BHs radiate. The entropy of a Schwarzschild BH is given by the Bekenstein–Hawking relation

$$S = \frac{\mathcal{A}}{4l_p^2}, \quad (1)$$

where  $\mathcal{A}$  is the area of the BH horizon and  $l_p = \sqrt{\frac{G\hbar}{c^3}}$  is the Planck length.

In the quest for a complete theory of quantum gravity (QG), several approaches to it have predicted particular forms for the QG-corrected BH entropy [6–15].

For example, starting from the quadratic generalized uncertainty principle (GUP) [7]<sup>1</sup>, whose effects can be implemented both in classical and quantum systems by defining deformed commutation relations by means of [17]

$$x_i = x_{0i} ; p_i = p_{0i} (1 + 2\alpha^2 p_0^2), \quad (2)$$

where  $[x_{0i}, p_{0j}] = i\hbar\delta_{ij}$  and  $p_0^2 = \sum_{j=1}^3 p_{0j}p_{0j}$  and  $\alpha = \alpha_0/m_p c$ , being  $\alpha_0$  a dimensionless constant, it is shown that the corrected BH entropy can be written as

$$S = \frac{\mathcal{A}}{4l_p^2} - \frac{\pi\alpha^2}{4} \ln\left(\frac{\mathcal{A}}{4l_p^2}\right) + \sum_{n=1}^{\infty} c_n \left(\frac{\mathcal{A}}{4l_p^2}\right)^{-n} + \text{const} \quad (3)$$

where  $c_n = \alpha^{2(n+1)}$ . We will take  $\hbar = c = G = 1$ . Therefore, within this choice,  $m_p = l_p = 1$  and  $\alpha = \alpha_0$ .

It is noteworthy that the Loop Quantum Gravity (LQG) prediction is obtained by considering  $\alpha = \sqrt{2/\pi}$ .

Recently, Scardigli and Casadio [18] proposed a deformed spherically symmetric and static Schwarzschild metric using the ansatz

$$f(r) = 1 - \frac{2M}{r} + \epsilon \frac{M^2}{r^2} \quad (4)$$

for the time–time component of the metric to reproduce the modified Hawking temperature as a consequence of the GUP. As pointed out very recently by A. F. Ali, M. M. Kahlil and E. C. Vagenas [19], this ansatz implies a different position for the event horizon, contrary to many arguments based on the GUP [20–22]. In Ref. [19], the authors extend the class of metrics which give place to the GUP-corrected Hawking temperature by assuming a functional dependence of the form

$$f(r) = \left(1 - \frac{2M}{r}\right) \left(1 + \eta \left(\frac{2M}{r}\right)^n\right), \quad (5)$$

where  $\eta \ll 1$  is a constant and  $n \geq 0$  is an integer. After comparing the modified Newton law with the Randall–Sundrum II model [23], the authors of [19] conclude that the most likely value for  $n$  is  $n = 2$ .

In this work we will look for a reinterpretation of the first and second terms of the RHS of Eq. (3) in a semiclassical way. Specifically, we will look for spherically symmetric and static geometries whose surface gravity at the horizon leads to the Bekenstein–Hawking *plus* the logarithmic correction Eq. (3). Thus, our approach tries to incorporate some GUP-related quantum gravitational effects in terms of geometries which satisfy Einstein’s equations.

In this sense, this work constitutes a semiclassical approach to the BH entropy. Interestingly, there have been also other works of semiclassical nature which try to solve the BH singularity problem by introducing modifications of the spherically symmetric Hamiltonian constraint in terms of holonomies (see, for example, [24–26] and references therein). Moreover, it is noteworthy that, although

\* p.bargueno@uniandes.edu.co

<sup>1</sup> The GUP gives rise to a minimal length scale which is thought to be a essential ingredient of any quantum gravitational theory [16].

the methods employed by the author of Refs. [24–26] are very different from ours, there are resemblances between some conclusions reached by these two approaches, as will be commented along the manuscript.

The paper is organized as follows. In section II, we briefly present how to derive BH geometries which include GUP effects on the entropy the Schwarzschild BH and the main properties of some of these geometries are analyzed in terms of certain deformations of the Schwarzschild solution. In section III, we will interpret the previous geometries in terms of gravity coupled to non-linear electrodynamics showing that our findings indicate the presence of a non-linear Reissner–Nordström (RN) BH. Finally, in section IV, a brief summary of the obtained results is given.

## II. SEMI-CLASSICAL METRICS FOR GUP-CORRECTED BLACK HOLE ENTROPY

The main idea is to obtain a spherically symmetric and static exact solution of Einstein’s equations such that its corresponding entropy, computed from the semi-classical gravity at the horizon, incorporates the GUP logarithmic correction. As commented along the Introduction, we demand that the location of the horizon of this proposed solution coincides with that of the Schwarzschild case, in agreement with many arguments based on the GUP [20–22]. Therefore, for a Schwarzschild-deformed metric of the form

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2d\Omega^2, \quad (6)$$

these requirements read  $\beta = \frac{2\pi}{\kappa}$ , where  $\kappa = \frac{f'(r_H)}{2}$  is the surface gravity at the horizon  $r_H = 2M$  and  $\beta$  is the inverse temperature of the BH. The prime denotes differentiation with respect to the radial variable.

If the ansatz function is taken to be of the form,

$$f(r) = \left(1 - \frac{2M}{r}\right)g(r) \quad (7)$$

after using the second law as  $dS = \beta dM$ , the following deformed-inverse temperature is obtained <sup>2</sup>

$$\beta = 8\pi M \left[1 - \left(\frac{\alpha m_p}{4M}\right)^2\right]. \quad (8)$$

Therefore, from the standard relations between  $\beta$  and  $\kappa$ ,  $g(r_H)$  reads

$$g(r_H) = \left[1 - \left(\frac{\alpha l_p}{2r_H}\right)^2\right]^{-1}. \quad (9)$$

Let us note that one possible choice for  $g(r)$  that satisfies the previous requirements gives place to a family of functions given by

$$g(r) = \left(1 - \frac{\alpha^2 l_p^2 (2M)^n}{4r^{n+2}}\right)^{-1}. \quad (10)$$

After a long but straightforward calculation, the algebraic curvature invariants reveal that there is an intrinsic singularity at  $r_s = \alpha l_p/2$  when  $n = 0$ . This means that the breakdown of classical general relativity occurs within a region whose length scale is the Planck length, as one should expect (we remind the reader that  $\alpha$  is a dimensionless constant of order unity). It is interesting to note that the  $S^2$  sphere of the LQG BH case [26] bounces on the minimum area of LQG and the singularity disappears. In our case, the singularity is still present, but this time near the Planck length. In the case of  $n \neq 0$  solutions there is also an intrinsic singularity at  $r_s = [\alpha^2 l_p^2 (2M)^n/4]^{1/(n+2)}$ . In these cases, the singular region depends not only on  $l_p$  but also on the mass  $M$  of the gravitating object. Therefore, we do not consider them as physically relevant as only the Planck scale is expected to be linked to the scale where quantum gravitational effects become dominant.

Thus, returning to the case  $n = 0$ , let us note that, considering the asymptotic behavior of  $f(r)$  we obtain

$$f(r) \longrightarrow 1 - \frac{2M}{r} + \frac{\alpha^2}{4r^2}. \quad (11)$$

Therefore, within this limit, the geometry can be interpreted as that of a deformed RN BH with an electric charge  $q$  such that  $\alpha = 2q$  (note that some similarities between the LQG and the RN BHs were pointed out in Refs. [24, 25] concerning mainly the causal structure of these spacetimes).

At this point, let us summarize our main findings:

- We have shown that the entropy associated to the deformed Schwarzschild metric corresponds to that of the first two terms of the RHS of Eq. (3).
- This deformed metric has an intrinsic singularity located at the Planck scale.
- At infinity, the metric behaves as that of a charged and static BH with  $\alpha = 2q$ .

Then, the next step is to look for a possible interpretation of the deformed metric, which we write as

<sup>2</sup> The Planck mass has been incorporated in order to have a dimensionally correct expression. In subsequent expressions, also the

Planck length (mass) will be sometimes incorporated to clarify the discussion.

$$ds^2 = - \left(1 - \frac{2M}{r}\right) \left(1 - \frac{q^2}{r^2}\right)^{-1} dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} \left(1 - \frac{q^2}{r^2}\right) dr^2 + r^2 d\Omega^2. \quad (12)$$

Given the previous RN-like interpretation at spatial infinity, it seems plausible to impose the geometry to be a solution to the Einstein–Maxwell system, when certain non-linear electrodynamics is invoked.

### III. COUPLING GRAVITY TO NON-LINEAR ELECTRODYNAMICS

By Israel’s theorem, the only electrovacuum static and spherically symmetric solution of the Einstein–Maxwell system is the RN one [27]. Therefore, Eq. (12) can not be a solution of this coupled system. However, our deformed Schwarzschild metric will appear when coupling gravity to a certain non-linear electrodynamics (NLED) theory. The importance of these theories is twofold: first, quantum corrections to Maxwell theory can be described by means of non-linear effective Lagrangians that define NLEDs as, for example, the Euler–Heisenberg Lagrangian [28, 29], which can be effectively described using Born–Infeld (BI) theory [30]. Second, it is well known that in case of dealing with open bosonic strings, the resulting tree-level effective Lagrangian is shown to coincide with the BI Lagrangian [31, 32]. Apart from gravitational BI solutions [33, 34], an exact regular BH geometry in the presence of NLED was obtained in [35] and further discussed in [36, 37]. In addition, the same type of solutions with Lagrangian densities that are powers of Maxwell’s Lagrangian were analyzed in [38]. Recently, a wide family of regular BHs satisfying the weak energy condition has been presented [39, 40].

Let us consider the following energy–momentum tensor for NLED:

$$T^{\mu\nu} = -\frac{1}{4\pi} [\mathcal{L}(F)g^{\mu\nu} + \mathcal{L}_F F^\mu{}_\rho F^{\rho\nu}], \quad (13)$$

where  $\mathcal{L}$  is the corresponding Lagrangian,  $F = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$  and  $\mathcal{L}_F = \frac{d\mathcal{L}}{dF}$ .

Assuming spherically symmetric and static electrovacuum solutions and taking only a radial electric field as the source, that is,

$$F_{\mu\nu} = E(r) (\delta_\mu^r \delta_\nu^t - \delta_\nu^r \delta_\mu^t), \quad (14)$$

Maxwell equations read

$$\nabla_\mu (F^{\mu\nu} \mathcal{L}_F) = 0. \quad (15)$$

Thus, from (14) and (15) one can obtain an explicit expression for the electric field

$$E(r) = -\frac{q}{r^2} (\mathcal{L}_F)^{-1}. \quad (16)$$

After some algebraic computations, the electric field is shown to be given by

$$E(r) = \frac{m'(r)}{q} - \frac{r}{2q} m''(r), \quad (17)$$

where the mass function  $m(r)$  is such that  $f(r) = 1 - 2m(r)/r$ .

In our case, using Eq. (12),  $m(r)$  results to be

$$m(r) = \frac{r(q^2 - 2Mr)}{2(q^2 - r^2)} \quad (18)$$

and the corresponding electric field is given by

$$\begin{aligned} E(r) &= \frac{q(q^4 + 2(5M - r)r^3 - q^2r(2M + 3r))}{2(q^2 - r^2)^3} \\ &= \frac{q}{r^2} + \mathcal{O}[r]^{-3}. \end{aligned} \quad (19)$$

Therefore, Eq. (12) behaves as a RN BH at infinity, which supports our description in terms of NLED.

The underlying NLED theory can be obtained using the  $P$  framework [41], which is somehow dual to the  $F$  framework. One introduces the tensor  $P_{\mu\nu} = \mathcal{L}_F F_{\mu\nu}$  together with its invariant  $P = -\frac{1}{4}P_{\mu\nu}P^{\mu\nu}$  and considers the Hamiltonian-like quantity

$$\mathcal{H} = 2F\mathcal{L}_F - \mathcal{L} \quad (20)$$

as a function of  $P$ . This quantity  $\mathcal{H}(P)$  specifies the theory. The Lagrangian can be written as a function of  $P$  as

$$\mathcal{L} = 2P \frac{d\mathcal{H}}{dP} - \mathcal{H}. \quad (21)$$

Finally, by reformulating the energy–momentum tensor in terms of  $P$ ,  $\mathcal{H}(P)$  is shown to be given by [37]

$$\mathcal{H}(P) = -\frac{1}{r^2} \frac{dm(r)}{dr}. \quad (22)$$

In our case, and considering only the Hamiltonian function for simplicity, the NLED can be shown to be given by

$$\mathcal{H}(P) = -\frac{P \left(1 + \sqrt{2Pq^2} - \frac{2^{5/4}P^{1/4}\sqrt{q}}{s}\right)}{1 + 2Pq^2 - \sqrt{8Pq^2}}. \quad (23)$$

where the parameter  $s = q/2M$  has been introduced to facilitate comparison with [35] (see the following discussion).

After Taylor expanding Eq. (23) we get

$$\mathcal{H}(P) = -P + \frac{2^{5/4}\sqrt{q}P^{5/4}}{s} - 3\sqrt{2q^2}P^{3/2} + \frac{2^{11/4}q^{3/2}P^{7/4}}{s} - 10q^2P^2 + \mathcal{O}[P]^{9/4}. \quad (24)$$

A couple of comments are in order here. First of all, let us note that Maxwell's theory,  $\mathcal{H}(P) = -P$ , is recovered for small fields. In addition, a quadratic BI-like term appears (fifth term in the RHS of Eq. (24)). This quadratic term is easy to interpret in light of the cutoff field which is an essential ingredient of BI-theory. Therefore, the difficulty of interpreting this NLED theory can

be ascribed to the other terms which appear in the RHS of Eq. (24). In spite of this, let us note that similar terms have appeared since the discovery of the first exact regular BH solution by Ayón-Beato and García [35]. In fact, the Hamiltonian function presented in Ref. [35] can be expanded for weak fields to give

$$\mathcal{H}_{AB}(P) = -P + \frac{3 \cdot 2^{1/4}\sqrt{q}P^{5/4}}{s} - 6\sqrt{2q^2}P^{3/2} + \frac{15q^{3/2}P^{7/4}}{2^{1/4}s} - 30q^2P^2 + \mathcal{O}[P]^{9/4}, \quad (25)$$

which except for some constants coincide with our Eq. (24).

We note that, although there are some similarities between the solution of Ayón-Beato and García and our Eq. (12), there are some essential differences between them, mainly concerning the weak energy condition (WEC), which states that the local energy density cannot be negative for all observers. For the metrics here considered, the WEC can be stated as

$$\begin{aligned} \frac{1}{r^2} \frac{dm(r)}{dr} &\geq 0 \\ \frac{2}{r} \frac{dm(r)}{dr} - \frac{d^2m(r)}{dr^2} &\geq 0. \end{aligned} \quad (26)$$

Therefore, it is easy to see that Eq. (12) violates the WEC. However, as can be shown by direct calculations, this violation is proportional to  $\alpha^2$ . Moreover, our solution can be considered perturbative in the following sense. We have considered corrections of order  $\alpha^2$  to the geometry which are compatible with a logarithmic correction to the BH entropy as predicted by a quadratic GUP. Furthermore, we have shown that the RHS of Einstein equations can be interpreted as some kind of NLED. The point is to note that also this NLED depends perturbatively on  $\alpha^2$  due to the fact that  $\alpha = 2q$ . Therefore, we are in a situation similar to that of Ref. [19] (where also the WEC is violated at order  $\alpha^2$ ) but in our case with a completely specified matter content which gives

place to the required entropy corrections.

#### IV. CONCLUSIONS

Although the search for a quantum theory of gravity is still under progress, some results about the behavior of the space-time at the Planck scale can be ascribed to the existence of a minimum length [16], which could be realized by a GUP which, among other implications, gives place to a logarithmic correction to the Bekenstein-Hawking black hole entropy. In this work we have proposed a deformation of the Schwarzschild metric which gives place to this logarithmic correction in a semiclassical way. This deformation preserves the location of the event horizon (as required by the GUP approach) and predicts the existence of a singularity at the Planck scale. Moreover, we have shown that this geometry is realized when gravity is coupled to a nonlinear electrodynamics model, obtaining an exact solution which has some resemblances with other well known regular black hole solutions. Although the weak energy condition is violated at second order in the GUP parameter (also reported in a recent work [19]), it would be interesting to investigate whether or not is possible to obtain effective geometries with reproduce the logarithmic correction without violating this energy condition. We hope to report on this in a future work.

P. B. acknowledges support from the Faculty of Science and Vicerrectoría de Investigaciones of Universidad de los Andes, Bogotá, Colombia.

- 
- [1] J. D. Bekenstein, Lett. Nuovo cimento **4**, 737 (1972).
  - [2] J. D. Bekenstein, Phys. Rev. D **7**, 2333 (1973).
  - [3] J. D. Bekenstein, Phys. Rev. D **9**, 3292 (1974).

- [4] S. W. Hawking, Nature **248**, 30 (1974).
- [5] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).

- [6] R. K. Kaul and P. Majumdar, Phys. Rev. Lett **84** , 5255 (2000).
- [7] A. J. Medved and E. C. Vagenas, Phys. Rev. D **70**, 124021 (2004).
- [8] G. A. Camelia, M. Arzano and A. Procaccini, Phys. Rev. D **70**, 107501 (2004).
- [9] A. Chatterjee and P. Majumdar, Phys. Rev. Lett. **92**, 141301 (2004).
- [10] M. M. Akbar and S. Das, Class. Quant. Grav. **21**, 1383 (2004).
- [11] Y. S. Myung, Phys. Lett. B **579**, 205 (2004).
- [12] A. Chatterjee and P. Majumdar, Phys. Rev. D **71**, 024003 (2005).
- [13] S. Das, P. Majumdar and R. K. Bhaduri, Class. Quant. Grav. **19**, 2355 (2002).
- [14] M. Domagala and J. Lewandowski, Class. Quantum Grav., **21**, 5233 (2004).
- [15] K. A. Meissner, Class. Quantum Grav., **21**, 5245 (2004).
- [16] S. Hossenfelder, Living Rev. Relativity, **16**, 2 (2013).
- [17] S. Das and E. C. Vagenas, Phys. Rev. Lett. **101**, 221301 (2008).
- [18] F. Scardigli and R. Casadio, Eur. Phys. J. C, **75**, 425 (2015).
- [19] A. F. Ali, M. M. Khalil and E. C. Vagenas, EPL **112**, 20005 (2015).
- [20] M. Maggiore, Phys. Lett. B **304**, 65 (1993).
- [21] G. Amelino-Camelia *et al.*, Class. Quantum Grav. **23**, 2585 (2006).
- [22] F. Scardigli, Phys. Lett. B **452**, 39 (1999).
- [23] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999).
- [24] L. Modesto, Class. Quantum Gravity **23**, 5587 (2006).
- [25] L. Modesto, gr-qc/0811.2196 (2008).
- [26] L. Modesto, Int. J. Theor. Phys. **49**, 1649 (2010).
- [27] W. Israel, Phys. Rev. **164**, 1776 (1967).
- [28] W. Heisenberg and H. Euler, Z. Phys. **98**, 714 (1936).
- [29] J. Schwinger, Phys. Rev. **82**, 664 (1951).
- [30] M. Born and L. Infeld, Proc. Roy. Soc. London A **144**, 425 (1934); *ibid.* **143**, 410 (1934) and **147**, 522 (1934).
- [31] E. S. Fradkin and A. A. Tseytlin, Phys. Lett B. **163**, 123 (1985).
- [32] A. A. Tseytlin, Nucl. Phys. B **501**, 41 (1997).
- [33] A. García D., H. Salazar I. and J. F. Plebański, Nuovo Cimento Ser. B **84**, 65 (1984).
- [34] N. Breton, Phys. Rev. D **67**, 124004 (2003).
- [35] E. Ayon-Beato and A. Garcia, Phys. Rev. Lett. **80**, 5056 (1998).
- [36] F. Baldovin, M. Novello, S.E. Perez Bergliaffa and J. Salim, Classical Quantum Gravity **17**, 3265 (2000).
- [37] K. A. Bronnikov, Phys. Rev. D **63**, 044005 (2001).
- [38] M. Hassaine and C. Martinez, Classical Quantum Gravity **25**, 195023 (2008).
- [39] Leonardo Balart and Elias C. Vagenas, Phys. Rev. D **90**, 124045 (2014).
- [40] Leonardo Balart and Elias C. Vagenas, Phys. Lett. B **730**, 14 (2014).
- [41] H. Salazar I., A. García D. and J. Plebański, J. Math. Phys. **28**, 2171 (1987).